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Towards an embodied, cultural, and material conception of mathematics cognition

Luis Radford

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Abstract In this paper I sketch an embodied, cultural, and material conception of cognition and discuss some of the implications for mathematics education. This approach, which I term sensuous cognition, rests on a cultural and historical dialectical materialist understanding of the senses, sensation, and the material and conceptual worlds. Sensation and matter are considered to be the substrate of mind, and of all psychic activity (cognitive, affective, volitional, etc.). I argue that human cognition can only be understood as a culturally and historically constituted multimodal sentient form of creatively responding, acting, feeling, transforming, and making sense of the world. To illustrate these ideas I briefly refer to a classroom episode involving 7- to 8-year-old students dealing with pattern generalization.

Keywords Cognition · Sensation · Senses · Materiality · Concepts · Multimodality

> Am I so dependent on the body and the senses that I could not be without them? Descartes (1641/1982, p. 19)

1 Introduction

In a Grade 10 class, working in small groups of three, 15- to 16-year-old students discuss how to draw a distance–time graph related to two moving individuals. The individuals (Mireille, to the left, and Nicholas, to the right; see Fig. 1) start walking at the same time on a straight line in opposite directions. Knowing that Nicholas starts from point R and stops at point S, while Mireille starts at point P and stops at point Q, the students are invited to come up with a graph conveying information about the distance between Mireille and Nicholas as time elapses.

Since no indication is given about speeds, in one of the groups, Zacko, Carla, and Jane start considering the simpler case—both individuals travel at the same speed. Distance, of course, decreases, but how? Carla suggests a kind of parabolic shape; Zacko suggests a straight line:

Carla: So she [Mireille] goes like this (referring to the drawing (Fig. 1) she moves her pen from Mireille's place towards Nicholas) … because she… she goes towards…
Zacko: Yeah, but he [Nicholas] is going towards her too… It's going to be weird…
Carla: OK, minute… a parabola!
Zacko: Its not a parabola! it would be a… steep [line] (making an almost vertical gesture)
Carla: Why?

Zacko then opens his arms and simulates the individuals' motion (see Fig. 2):

Zacko: I am just coming closer and closer in less time (while talking, he moves slowly his hands one towards the other).
Jane: Yes, like… it takes less time to arrive… (Articulating once more her thought) because it has taken half the time to arrive.

About two decades ago, an analysis of the previous episode for a mathematics education audience would have

L. Radford (✉)
Université Laurentienne, Sudbury, Canada
e-mail: lradford@laurentian.ca

L. Radford
The University of Manchester, Manchester, UK

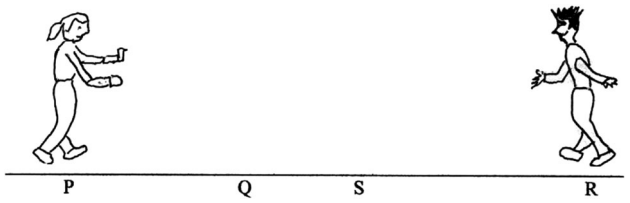

Fig. 1 Mireille (*left*) and Nicholas (*right*) move towards each other

been unlikely to include an analysis of gestures, body posture, and other embodied signs. Indeed, most of the psychological traditions that inspired our field at the time conceptualized the body and material culture as playing a secondary role in cognition. And if the body played some role, as in Piaget's (1970) influential genetic epistemology, it was as a mere transitory step towards abstract, genuine thinking. That was the fate of the sensory-motor stage in Piaget's account of conceptual development.

New research trends, however, offer a different approach to the understanding of human cognition. They consider our tactile-kinesthetic bodily experience of the world and our interaction with artifacts to be much more than transitory or merely secondary aspects of cognition (Bautista and Roth 2012; Borba and Villareal 2006; Edwards et al. 2009; Lakoff and Núñez 2000; Sheets-Johnstone 2009). Within these research trends, an account of the students' understanding of the problem and the production of their graph in the previous episode would require an analysis of the students' embodied and discursive activity. Many interpretations and accounts are possible. For instance, some approaches inspired by cognitive linguistics (e.g., Fauconnier and Turner 2002; McNeill 2005) emphasize the metaphoric dimension of language and the integrative constitution of embodied mental spaces (see e.g., Edwards 2009; Lakoff and Núñez 2000; Yoon et al. 2011). Other approaches inspired by research in phenomenology emphasize the fleshy nature of thought (Thom and Roth 2011), while others stress its material dimension (de Freitas and Sinclair 2013).

Embodied and materialist approaches to cognition, however, face the difficult problem of offering a cogent account of the theoretical categories of the conceptual and the embodied, and their relationships. The apparent absence of compelling accounts of these categories has led some critics, such as Terry Eagleton, to see in contemporary embodied perspectives of the mind no more than "the return in a more sophisticated register of the old organicism" promoted by seventeenth century empiricists and a token of "the post-modern cult of pleasure" and love for the concrete (Eagleton 1998, pp. 157–158). What Eagleton fears is that contemporary embodied perspectives remain trapped in the sensing subject, and hence in a curious form of radical subjectivism with unclear possibilities to regain contact with culture and history (Eagleton 1996).

Dwelling upon Vygotsky's (1987–1999) and Leont'ev's (1978, 2009) work, and Hegelian dialectic materialism (Ilyenkov 1977), in this paper I articulate a cultural-historical theoretical perspective on cognition. Briefly put, in the perspective on cognition that I outline here—that I have previously termed sensuous cognition (Radford 2009a)—mind, body, and world are conceived of as intertwined entities. Sensuous cognition stresses the idea that our thinking, feelings, deeds, and in fact all our relations to the world (hearing, perceiving, smelling, sensing, etc.), are an entanglement of our body and material and ideational culture.

The idea of sensuous cognition rests on a specific historical understanding of sense, sensation, materiality, and the conceptual realm. Within this theoretical perspective, our cognitive domain can only be understood as a culturally and historically constituted sentient form of creatively responding, acting, feeling, imagining, transforming, and making sense of the world. I articulate this idea in the second part of the paper. In the first part, I discuss how idealist and empiricist epistemologies framed the problem of the sensual and the conceptual. In the third part of the paper, to illustrate in a concrete manner the idea of sensuous cognition, I briefly refer to a classroom episode

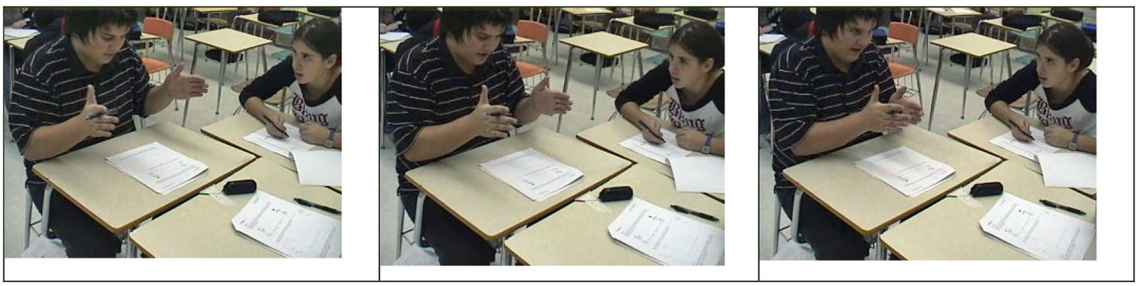

Fig. 2 Zacko makes a dynamic gesture to signify the decreasing space between Mireille and Nicholas

involving 7- to 8-year-old students dealing with pattern generalization.

2 The sensual and the conceptual

The insouciance that classical psychology has shown towards the body in the investigation of human thinking is related to a great extent to the idealist theories of knowing that have been influential since Plato's time.

Plato considered the body to be a nuisance or obstacle in the attainment of true knowledge (Radford 2005; Radford et al. 2009). The seventeenth and eighteenth century idealists continued this tradition. When, in Second Meditation, Descartes asks the question: "What am I?" he answers: "A thinking substance" (Descartes 1641/1982, p. 22). "I am anything but mind" (p. 25). For Descartes, to know something amounted to having a distinct apprehension of the thing to be known. "I cannot be deceived in judgments of the grounds of which I possess a clear knowledge" (p. 56). And apprehension and the distinctiveness of things were not ensured by the senses. Thus, to explain how bodies and external things become known, Descartes says that "bodies themselves are not properly perceived by the senses nor by the faculty of imagination" (Descartes 1641/1982, p. 26). True knowledge is ensured, Descartes continues, "by the intellect alone … [things] are not perceived because they are seen and touched, but only because they are rightly comprehended by the mind" (p. 26). Knowledge was not to be sought in the materiality of things or in the feelings of the sentient body. Knowledge and ideas, the idealist philosophers supposed, are in us. This is what Leibniz contended: "our ideas, even those of sensible things, come from within our own soul" (Leibniz 1705/1949, p. 15).

However, not all philosophical traditions followed a disembodied view of thinking and knowing. The empiricists offered an opposing view. Hume (1748/1921), for instance, argued that ideas are impressions or combinations of impressions that external things cause. In a passage of An Enquiry Concerning Human Understanding, he notes that although

> our thought seems to possess this unbounded liberty, we shall find, upon a nearer examination, that it is really confined within very narrow limits, and that all this creative power of the mind amounts to no more than the faculty of compounding transposing, augmenting, or diminishing the materials afforded us by the senses and experience. (Hume 1748/1921, p. 16)

Kant tried to articulate a theory of knowledge that combines the idealist and the empiricist tenets, resulting in a kind of compromised rationalism traversed by unresolvable tensions. In the Critique of Pure Reason Kant says:

> A new light flashed upon the mind of the first man (be he Thales or some other) who demonstrated the properties of the isosceles triangle. The true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what was necessarily implied in the concepts that he had himself formed a priori, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with a priori certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept. (Kant 1781/1929, p. 19, B xi–xii)

The first part of the passage reveals Kant's acknowledgment and dismissal of the British empiricist approach to knowledge formation—as elaborated, in particular, by Hume (1748/1921) and Berkeley (1710/1957). The second part ("but to bring out…") reveals Kant's alignment with the rationalist tradition (as epitomized by Descartes, Leibniz, and others). In fact, Kant's epistemology is a desperate battle to reconcile sensuous empiricism and conceptual idealism. As the example of the triangle suggests, according to Kant, in coming to know something, we resort to a *sensuous construction or representation*. However, what we represent is not *the* thing to be known (for instance, the concept of triangle) but *a* particular one. Furthermore, what we learn of the thing to be known is not read from the particular representation thus drawn, but from something *general* that, for him, is beyond the sensuous realm: something not sensual but intellectual.

Kant's epistemology rests indeed on the claim that human cognition will always need *both* a sensible component and an intellectual one. He theorizes the sensible part through the concept of *sensibility*: that is, our capacity for being affected by material things. He theorizes the intellectual component through the concept of *understanding*. He says:

> Without sensibility no object would be given to us, without understanding no object would be thought. Thoughts without content are empty, intuitions [i.e., representations] without concepts are blind… These two powers or capacities [sensibility and understanding] cannot exchange their functions. The understanding can intuit nothing, the senses can think nothing. Only through their union can knowledge arise. (Kant 1781/1929, p. 193, B75–76)

To remain faithful to the rationalist project, Kant hence ended up portraying a theory where reason appears supported by the concrete realm of the senses and where, in the end, the rationalist component nevertheless predominates.

He can offer a view of reason that, while requiring a sensual support, obeys universal logical mechanisms regardless of culture and time. Piaget followed the same path—removing nevertheless all the aprioristic stance of Kant's theory of knowledge. The sensory-motor stage yields the developmental road of embodied actions to disembodied intellectual thought. The ephemeral concrete realm of action and materiality remains the carrier and the expression of a thinking measured by its rational structural features.

The previous discussion provides us with a short overview of the manner in which idealist epistemologies envisioned the realm of the conceptual and the sensual. Certainly, Kant offers an interesting and profound account. His concepts of sensibility and understanding are an attempt to respond to the sensual exigencies of empiricism and the logical demands of rationalism, respectively. In doing so, however, he adopts an intrinsic dualism that keeps the conceptual and the sensual separately.

Hegel (1830/2009) criticized Kant's dualistic stance. The problem, Hegel considered, was that Kant adopted an impoverished idea of the conceptual and the sensual, and an unnecessary distinction between them. In a passage of the Phenomenology of Spirit, he noted that "It is a mere confusion when … being-in-itself (the Kantian noumenon) is postulated as being beyond being-for-thought" (Hegel 1807/1977). In other words, it is a mistake, Hegel contended, to argue, as Kant did, that things in themselves (e.g., a chair or a circle not as material things, but as things independent of all experience) are unknowable, beyond human cognition. For Hegel, Kant's problematic assumption rests in the demarcation that he drew from the outset between the sensuous experience of the empirical world and the world of concepts. As Kant understood human cognition, concepts and objects remain irremediably separated: they are two heterogeneous parts of cognition. Thus, in the Critique of Judgment, Kant (1790/2005) portrays concepts as the extended possibility of sensuous objects and sensuous objects as the mere raw material for cognition. In Hegel's view, this separation between the sensual and the conceptual leads to a weak concept of the concrete world and subjectivity. On the one hand, the world is there, as a mere raw material of experience; that is, as a repository of stimuli—no more, no less. On the other hand, subjectivity is reduced to the passive reception of stimuli and the rational universal cognitive mechanisms that the raw sensuous material sets into motion to subsume objects into concepts.

In the next section I sketch a view of the human mind that draws from a different philosophical perspective—the Hegelian historical-materialist dialectic paradigm—that posits mind and matter as consubstantial entities. Such a view provides, I argue, new possibilities to understand the role of senses, sensation, and materiality in the teaching and learning of mathematics.

3 Sensuous cognition

In dualistic accounts, the mind is conceived of as operating through two distinctive planes, one internal and one external. The internal plane is usually considered to include consciousness, thought, ideas, intentions, etc., while the external plane refers to the material world—which includes concrete objects, our body, its movements, and so on. The idea of sensuous cognition that I advocate here rests on a non-dualistic view of the mind. Within this theoretical view, mind is considered to be a property of matter. More specifically, mind is conceptualized as a feature of living material bodies characterized by a capacity for responsive sensation.

Sensation is a phylogenetically evolved feature of living organisms through which they reflect and respond to or act on their environment. Since the organism is itself a part of the material world, any reflection of and action on reality is strictly a function of a material, corporeal organism (Leont'ev 2009). As a result, reflection and action do not occur in two separate planes. They occur in the same plane—the plane of life.

Now, reflection cannot be considered a passive act of receiving sensorial impressions, as seventeenth and eighteenth century empiricists hold. Reflection involves both: (1) something that transcends the organism as such (something that, in order to differentiate it from the subject itself, we can call objective, namely the object of reflection), and (2) the reflected object, something that is subjective (in the sense that reflection depends on the specific organism reflecting the environment). In this account, reflection is a relational category between subject and object that keeps them entangled with each other. Reflection's phylogenetic evolution gives rise to a range of sensuous relationships between the organism and the world (thermal and other forms of sensation, movement, and so on).

In the case of humans, the entanglement of subject and object, mind and world, acquires a specific dimension by virtue of their historically evolved capacity to interact with the world and among species' members. The explanations of such a capacity is beyond the scope of this paper (see, e.g., Greenspan and Shanker 2004; Leont'ev 2009; Mithen 1996). Suffice it to say that this evolution culminated with a progressive refinement of the senses and sensation. This is why the forming of the human senses is not a natural process but the "labor of the entire history of the world down to the present" (Marx 1932/1988, pp. 108–109). The human eye, for instance, "has become a human eye, just as

its object has become a social, human object" (p. 107), that is an object of labor, emanating from social sensuous interaction. In the course of labor and interaction, the human senses appeared hence transformed and became what they are now, highly sensitive cultural organs—"*theoreticians*" as Marx put it (1932/1988, p. 107, emphasis as in the original).

The historical nature of sense and sensation and the entanglement of cognition and the world make it impossible to reduce the material world to pure stuff. Matter, notwithstanding the empiricists, is not merely the concrete stuff that we touch with our hands or perceive with our eyes. Matter—all that is revealed to us in our dealing with the world—is cultural-historical in the sense that it bears in itself the traces of human labor and intellectual activity. To see matter subjectively only is to make the mistake that Marx reproached to materialist philosopher Ludwig Feuerbach (1843/1986):

> [Feuerbach] does not see that the sensuous world around him is not a thing given direct from all eternity, remaining ever the same, but the product of industry and of the state of society; and, indeed, [a product] in the sense that it is an historical product, the result of the activity of a whole succession of generations… Even the objects of the simplest "sensuous certainty" are only given him through social development, industry and commercial intercourse. (Marx 1846/1998, p. 45)

What Feuerbach does not grasp, Marx contends, is that what we access through our senses is not the world in its intact materiality, Adam and Eve's intact world, but a world of labor replete with historical and cultural sensuous activity:

> The cherry-tree, like almost all fruit-trees, was, as is well known, only a few centuries ago transplanted by commerce into our zone, and therefore only by this action of a definite society in a definite age it has become "sensuous certainty" for Feuerbach. (Marx 1846/1998, p. 45; emphasis in the original)

The concept of sensuous cognition rests on this understanding of sense, sensation, and the material world as historical and cultural products.

4 Revisiting the conceptual

The aforementioned emphasis on sensation and matter does not amount, however, to reducing the mind to the realm of pure senses or the materiality of the world. The chief idea of sensuous cognition is that our thinking, feelings, deeds, in fact all of relations to the world (hearing, perceiving, smelling, sensing, etc.), are an entanglement of both the material world and ideational culture.

Now, the notion of ideational culture rests on a dialectical understanding of the conceptual. In classical rationalist and idealist epistemologies the conceptual is reduced to something already given—concepts are mind-independent entities (e.g., Kant's (1781/1929, p. 23) "things-in-themselves")—or are the products of the individual's cogitations. Western philosophy is to an important extent the tormented effort to explain how we can have access to mind-independent entities or, conversely, how the subject's self-produced ideas relate to the concrete world. In one case or the other, concepts are considered susceptible to be represented. They are somewhere "there" (in a transcendental world of ideas or inside the subject's mind). To the static ontology of Kant, the Platonists, and the Idealists, Hegel opposed an ontology of movement. Concepts (which belong to Hegel's theoretical category of the general) are pure virtuality or possibility. They come into existence when we set them in motion, in practical embodied and material activity, and through which they appear in sensuous singularity. When Vygotsky (1987) argues that all conceptual phenomena need to be studied as processes in motion and change, he is drawing on this Hegelian feature of concepts.

Concepts considered as pure possibility are not to be confused with their idealist or rationalist counterparts. Thus, instead of being a "a representation which is contained in an infinite number of different possible representations (as their common character)" (Kant 1781/1929, pp. 69–70), concepts in the Hegelian dialectic materialism are culturally codified historical forms of human action continuously affected and transformed by social practices. The concept of number, for instance, is a pure possibility (e.g., to count things or to carry out complex calculations). To come into existence, concepts have to be endowed with particular determinations. Their coming into existence is the event of their appearance—an always new and unrepeatable sensuous and material event, simultaneously historical and actual, in which the abstract and the concrete, the sensual and the conceptual, come together and stand to each other in a relation of mutual determination. Both are different while at the same time being part of a same organic whole-and-part unity. Through their particular sensuous instances concepts are not transcendentally inaccessible. They are entities that move and are continuously transformed by the activity through which they appear in their sensuous and material form. They cannot appear by themselves: they are mediated entities and their mediation is concrete human practice (Radford 2013a).

5 The multimodal nature of concepts

Human action is multimodal. As codified historical forms of human action, concepts are multimodal too. But they are also multimodal in their actualization, in the passage from the virtual to the actual. Indeed, in their movement into existence, in which they become objects of thought and consciousness, concepts are endowed with particular determinations. They have to be actualized in sensuous multimodal and material activity.

So, in our introductory example, when Zacko opens his arms and puts his hands at a certain distance to simulate Mireille's and Nicholas' walk, he draws on, and puts into motion, cultural-historical concepts of space and time. And he does so according to his own understanding of those concepts—understanding that is built on his prior formal and informal acquaintance with them. The distance between Mireille and Nicholas that Zacko simulates is not accurate, nor is their speed. Yet, through the sensuous enactment of the walk, Zacko feels the distance decrease. His eyes (and his teammates' eyes too) are fixated on the hands. Not on the hands as such, but on the hands as means to sensuously imagine the phenomenon under consideration. Zacko follows perceptually and viscerally each hand moving towards the other, resorting at the same time to something that apes, insects, and other species lack: a historically constituted and highly categorial language. Through language Zacko can qualify what the body in itself can feel but cannot objectify beyond the lived present and kinesthetic memory (Freitag 2002): an experience that attends in a categorial manner to the quality of proximity of the moving individuals and the effect of time. While moving and feeling his hands, Zacko says: "I am just coming closer and closer in less time." In the enactment of the concept, during the concept's transition from pure possibility to actuality, the concept appears in its multimodal actuality. The sensual and the conceptual become entangled. No distinction between them is possible. Distance and time—two culturally and historically evolved conceptual categories (Radford 2008)—become entangled with Zacko's sensuous bodily multimodal actions.

6 The ontogenetic development of the senses

In the previous sections I outlined a reconceptualization of the sensual and the conceptual. This theoretical reconceptualization offers new avenues to understand cognition as sensuous through and through. It also allows us to posit the problem of the development of cognition as a problem intimately related to the cultural development of the senses and multimodality. Indeed, when Zacko opens his arms and moves his hands, one towards the other, he is resorting to an ontogenetically evolved form of multimodality in which the various modes of sensation have become specialized and integrated. Research with newborns shows that, at birth, sensorial modalities are not equally active (Lickliter and Bahrick 2000). While tactual sensitivity, for instance, is highly developed in the newborn around the mouth–nose cavity, spreading throughout other areas of the body surface later on, olfactory sensitivity is minimal (Zaporozhets and Elkonin 1971). The third month of life is crucial, for it is in this period that intersensory connections are formed and that the feeling movements of the hand appear. In the fifth and sixth months of life an enlargement and specialization of the development of intermodal connections happens in the visual-kinesthetic and visual-tactual spheres. Thus, "turning the head toward the point of contact [of object and body] and visually fixating upon that point, are expressive of visual-tactual connections" (Yendovitskaya et al. 1971, p. 15; see also Lewkowicz and Lickliter 1994). The specialization of the senses and the acquisition of language will ensure later on that, through a tactile experience, the human hand does not only feel the trace of the object. We can say that the hand also "perceives its colour, its volume, its weight" (Le Breton 2007 p. 151).

In short, like tactility, cognitive functions such as thinking, memory, and imagination remain directly and indirectly related to the materiality and conceptuality of the world (as conveyed, e.g., through linguistic conceptual categories, such as "distance" and "space" in our example) and to a large range of sensorimotor evolved functions expressed through the organism's movement, perception, sound reception and production, etc. Our cognitive functions unfold and evolve in correlation with the cultural transformation of our senses and our multimodal sensorial experience of the world.

The ontogenetic process of the cultural transformation of the senses has been investigated in great detail in the past few years. To mention but one example, Zaporozhets (2002) reports research with 3- to 5-year-old preschoolers who were learning to discriminate between variants of two geometric figures: triangles and quadrilaterals. At first, the preschoolers were making a substantial number of errors. Then, they were invited to trace systematically with a finger the outline of the figure, paying attention to directional changes of the motions at angles, and accompanying the tactile exploration with side counting (1, 2, 3…). The investigator reports that at this stage perception was accomplished through the tactile experience, while the eye performed an auxiliary role. "Later," Zaporozhets says, "the eye developed the ability to solve these types of perceptual tasks independently, consecutively tracing the outline of a figure, as it was earlier done by a touching hand" (2002, p. 31). During this process, the eye undergoes

a transformative change: "initially, the eye motions have an extremely extensive nature, consecutively tracing the entire outline of the perceived figure and simulating its specifics in all details" (p. 32). In a subsequent stage, the eye's motions "gradually begin to decrease and to focus on the individual, most informative attributes of the object" (p. 32).

7 The artifactual dimension of sensuous cognition

A closer look at the previous example shows that the new cultural forms of sensation are deeply interrelated with the use of artifacts. In Zaporozhets' (2002) example, preschool children develop a mathematical form of perception that allows them to distinguish between cultural categories of geometrical figures. In the course of this developmental process, the children have recourse to the material objects whose contours they cover with a finger while using numbers to count aloud. What this example shows is that our individual senses evolve intertwined not only one with the other senses, but also with the materiality of the objects in our surroundings. The materiality that shapes our senses is not, however, reduced to inert matter, but, as the example shows, matter already endowed with meaning (e.g., "triangularity," "quadrilarity," etc.).

It is this key role of artifacts in the constitution and evolution of forms of sensing and reflecting that Luria and Vygotsky underlined in their work. The use of artifacts, they contended, constitutes the first phase in cultural development (Luria and Vygotsky 1998; Vygotsky and Luria 1994). Such a phase marks the emergence of new forms of actions and reflection and the concomitant appearance of psychic functions.

All in all, the previous discussion amounts to making a point about the embedded nature of artifacts in the evolution of our ways of sensing and reflecting. Luria and Vygotsky stress the fundamental cognitive role artifacts and material culture play in the ways we come to know. The claim that I am making hence goes beyond the conceptualization of artifacts as merely *mediators* of human thinking and experience, or as prostheses of the body. Artifacts do much more than mediate: they are a *constitutive part* of thinking. Behind this view lies, of course, the general concept of mind as a property of matter stated at the beginning of Sect. 3. This property expresses the enactive relationship between materiality and mind that inspired Vygotsky's, Luria's, and Leont'ev's work and that Bateson (1973) illustrates so nicely in his example of the blind person's stick. It is in this context that anthropologists Malafouris and Renfrew (2010) claim that we can speak of things as having a cognitive life. They say: "things have a cognitive life because minds have a material life" (p. 4).

Sensuous cognition is thus a perspective that highlights the role of sensation and materiality as the substrate of mind and of all psychic activity. But in contrast to other approaches where the focus remains on the individual's body, sensuous cognition offers a perspective where sensation and its cultural transformation in sensing forms of action and reflection are understood to be interwoven with history, cultural artifacts, and materiality at large. Sensuous cognition calls into question the usual divide between mind and matter and casts in new terms the classical boundaries of mind. It offers a new perspective in which to conceive of students' and teachers' actions in teaching–learning processes. In particular, sensuous cognition invites us to pay attention to perception, gestures, kinesthetic actions, sign- and artifact-use in different ways.

8 A classroom example

In what follows, I would like to discuss an example from a Grade 2 class (7- to 8-year-old students) involving the generalization of an elementary figural sequence (see Fig. 3).

To become sensitive to the cultural-historical algebraic forms of perceiving terms in sequences such as the one discussed here, students engage in processes that are far from mental. They engage with the task of exploring the sequence in a sensuous manner. I would like to illustrate this point by discussing the way in which the teacher and a group of students reflect on Term 8 of the sequence. The first question of the mathematical activity consisted in extending the terms of the sequence up to Term 6. Then, in questions 2 and 3, the students were asked to find out the number of squares in Terms 12 and 25. In question 4, the students were given a term that looked like Term 8 of the sequence (see Fig. 4). They were told that this term was drawn by Monique (an imaginary Grade 2 student) and

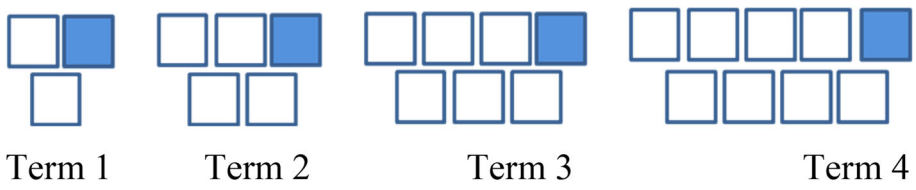

Fig. 3 The first terms of a sequence that 7- to 8-year-old students investigate in a Grade 2 class

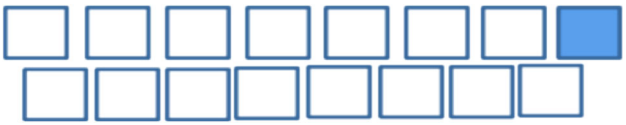

Fig. 4 The students were requested to discuss whether Monique's term is Term 8 of the given sequence

encouraged to discuss in small groups to decide whether or not Monique's term was Term 8. The trained eye would not have difficulties in noticing the missing white square on the top row. The untrained eye, by contrast, may be satisfied with the apparent spatial resemblance of these terms with the other terms of the sequence and might consequently fail to note the missing square.

Let me focus on the discussion that a group of students had with the teacher—a group formed by James, Sandra and Carla. When the teacher came to see their work the students had already worked for about 32 min together. They had finished drawing Terms 5 and 6, tried (unsuccessfully) to find the number of squares in Terms 12 and 25, and answered the question about Term 8 (which they considered to be Term 8 of the sequence). The teacher engaged in collaborative actions to create the conditions of possibility for the students to perceive a general structure behind the sequence:

1. Teacher: We will just look at the squares that are on the bottom (while saying this, the teacher makes three consecutive sliding gestures, each one going from bottom row of Term 1 to bottom row of Term 4; Pics 1–2 in Fig. 5 show the beginning and end of the first sliding gesture). Only the ones on the bottom. Not the ones that are on the top. In Term 1 (she points with her two index fingers to the bottom row of Term 1; see Pic. 3), how many [squares] are there?

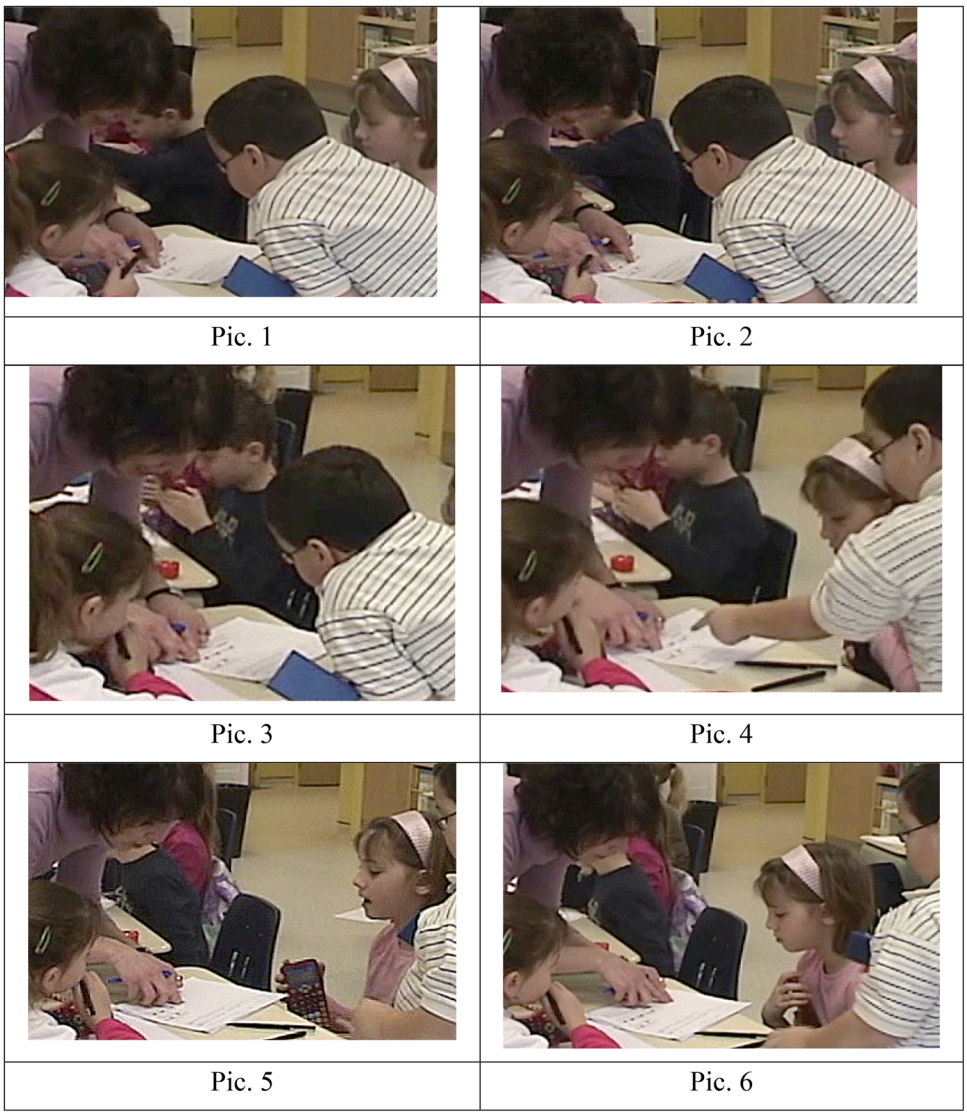

Fig. 5 The teacher's and students' sensuous (perceptual, gestural, tactile, aural, vocal) engagement in the task

2. Students: 1!
3. Teacher: (Pointing with her two-finger indexical gesture to the bottom row of Term 2) Term 2?
4. Students: 2! (James points to the bottom row of Term 2; see Pic. 4).
5. Teacher: (Pointing with her two-finger indexical gesture to the bottom row of Term 3) Term 3?
6. Students: 3!
7. Teacher: (Pointing with her two-finger indexical gesture to the bottom row of Term 4; see Pic. 5) Term 4?
8. Students: 4!
9. Teacher: (Making a short pause and breaking the rhythmic count of the previous terms, as if starting a new theme in the counting process, she moves the hand far away from Term 4 and points with a two-finger indexical gesture to the hypothetical place where one would expect to find Term 8; see Pic. 6) How many squares would Term 8 have on the bottom?
10. Sandra: (Hesitantly, after a relatively long pause) 4?

In Line 1, the teacher makes three sliding gestures to emphasize the fact that they will count the bottom row of the four given terms. The gestural dimension of the teacher–students' joint activity is somehow similar to the tactile experience of the students who, in the aforementioned experiment reported by Zaporozhets (2002), follow the contour of shapes with their fingers. Here, the tactile dimension is carried out instead with gestures through which the teacher suggests a cultural form of perceiving the terms of the sequence—one in which the mathematical ideas of variable and the relationship between variables are emphasized.

Now, the teacher does not gesture silently. Gestures are coordinated with utterances. This is why it might be more useful to consider the teacher's utterance as a multimodal utterance, that is to say as a bodily expression that resorts to various sensorial channels and different semiotic registers (Arzarello 2006). In this case, the teacher coordinates eye, hand, and speech through a series of organized simultaneous actions that orient the students' perception and emergent understanding of the target mathematical ideas. In our previous work we have termed semiotic node this complex coordination of various sensorial and semiotic registers (Radford 2009b). The investigation of semiotic nodes in classroom activity is a crucial point in understanding the students' learning processes. The concept of semiotic node rests indeed on the idea that the understanding of multimodal action does not consist in making an inventory of signs and sensorial channels at work in a certain context. From a methodological viewpoint, the problem is to understand how the diverse sensorial channels and semiotic signs (linguistic, written symbols, diagrams, etc.) are related, coordinated, and subsumed into a new thinking or psychic unity (Radford 2012). Such a methodological problem makes sense only against the background of a conception of the mind that overcomes the dualistic view of internal–external processes. In our case, the methodological problem makes sense against the background of a concept of the human mind as sensuous through and through.

Yet, as Line ten intimates, the passage from Term 4 to Term 8 was not successful. The objectification (that is, the becoming aware; see Radford 2010) of the algebraic manner in which sequences can be algebraically perceived has not occurred yet. The teacher hence decided to restart the process, with some important modifications, as we shall see.

As mentioned previously, Term 8 of the sequence was not materially drawn. In the previous excerpt, the teacher pretends that Term 8 is on the empty space of the sheet, somewhere to the right of Term 4. She points to the empty space, as she pointed to the other terms, to help the students imagine the term under consideration. During the second attempt, the teacher does not go from Term 4 to Term 8; this time she goes term after term until Term 8:

11. Teacher: We will do it again…
12. Teacher: (Pointing to Term 1 with a two-finger indexical gesture) Term 1, has how many?
13. Carla: (Pointing with her pen to the bottom row) 1, (without talking to the teacher points to Term 2 with a two-finger indexical gesture; Carla points with her pen to the bottom row of Term 2) 2, (again without talking to the teacher points to Term 3 with a two-finger indexical gesture; Carla points with her pen to the bottom row of Term 3) 3, (same as above) 4, (now moving to the hypothetical place where Term 5 is expected to be and doing as above) 5.
14. Teacher: Now it's Term 8! (The teacher comes back to Term 1. She points again with a two-finger indexical gesture to the bottom row of Term 1) Term 1, has how many [squares] on the bottom?
15. Students: 1.
16. Teacher: (Pointing with a two-finger indexical gesture to the bottom row of Term 2) Term 2?
17. Students: 2!
18. Teacher: (Pointing with a two-finger indexical gesture to the bottom row of Term 3) Term 3?
19. Students: 3!
20. Teacher: (Pointing with a two-finger indexical gesture to the hypothetical place where the bottom row of Term 4 would be) Term 4?
21. Students: 4!
22. Teacher: (Pointing as above) Term 5?
23. Students: 5!

24. Teacher: (Pointing as above) Term 6?
25. Students: 6!
26. Teacher: (Pointing as above) Term 7?
27. Students: 7!
28. Teacher: (Pointing as above) Term 8?
29. Students: 8!
30. Sandra: There would be 8 on the bottom!

The teacher and the students counted together the squares on the bottom row of Monique's term and realized that the number was indeed 8. At this point the relationship between variables started becoming apparent for the students. The relationship started being objectified. The teacher then moved to a joint process of counting the squares on the top row. The students were perplexed to see that contrary to what they believed, Monique's Term 8 did not fit into the sequence (for details, see Radford 2010).

Later on in the lesson the students were able to quickly answer questions about remote terms, such as Term 12 and Term 25, which were not perceptually accessible. They refined the manner in which the terms of the sequence could be perceived. The number of squares on the bottom row was equated to the number of the term in the sequence, while the number of squares on the top row was equated to the number of the term plus one. Here is an excerpt from the dialogue of Sandra's group as they discuss without the teacher:

31. Sandra: (Referring to Term 12) 12 plus 12, plus 1.
32. Carla: (Using a calculator) 12 plus 12 … plus 1 equal to …
33. James: (Interrupting) 25.
34. Sandra: Yeah!
35. Carla: (*Looking at the calculator*) 25!

Through an intense interplay between various sensorial modalities and different signs, the students' perception and the concomitant mathematical thinking have gained a theoretical dimension that they did not have before. The interplay of the various senses has become contracted, subsumed, and reorganized into a new complex psychic unity where no reference is made to top or bottom rows. Relational aspects of the variables in play are now emphasized. As a result, the students are now capable of dealing in a quick manner with remote terms.

The transformation of the students' initial perception into a new theoretical form of perception and thinking rests on a historically constituted manner of perceiving things that started with the Pythagorean and neo-Platonist investigations of figural numbers and patterns in Antiquity and that was continuously refined during cultural evolution (see Radford 2006, 2013a for details). As an ideal form of seeing, this theoretical perception appeared, from the students' viewpoint, as mere potentiality or mere virtuality. Through classroom activity and its pedagogical design, the ideal perceptive form was set into motion. Actualized through sensuous and material activity, it became an object of sensuous material thought and consciousness for the students.

9 Synthesis and concluding remarks

In this paper, I argued that one of the challenges to contemporary embodied and materialist approaches to human cognition consists in overcoming the dualism between the sensual and the conceptual that is a chief characteristic of empiricist and rationalist epistemologies. I opened the paper by briefly discussing some features of Descartes' and Kant's work. I noted that, by maintaining an irreducible gap between the sensual and the conceptual, Kant's concepts remain phenomenologically inaccessible. It is only by an allegedly aprioristic architecture of the human mind that we can subsume the material and empirical into the conceptual. In order to move beyond the transcendental idealism that characterizes Kant's theory of knowing, I claimed, it is necessary to rethink the conceptual and the sensual and their relationships. I sought support in Hegel's philosophy. In his criticism of Kant's work and idealism in general, Hegel argued that a paramount problem is the heterogeneity idealists assume concerning the conceptual and the sensual. Hegel contended that we need to resist considering that the conceptual and the sensual "are mutually external to each other" (Hegel 1801/1977, p. 165) and suggested that between them there is rather a mutual—or dialectical—constitution.

Drawing on Vygotsky's (1987–1999) and Leont'ev's (1978, 2009) work and Hegel's (1830/2009) dialectic materialism, I sketched a theoretical perspective on cognition. This theoretical perspective rests on a cultural-historical understanding of sense, sensation, and the material world. It considers matter and sensation as the substrate of mind and all psychic activity. From this theoretical perspective emerges a sensuous concept of cognition where, instead of being something purely "mental," thinking, reflecting, imagining, etc. remain intertwined with the material and conceptual aspects of the world and with the organism's capacities for sensation.

However, the concept of sensuous cognition goes beyond asserting that mind, body, and world are dialectically intertwined entities. It also asserts that *cognition is transformed by human labor and social practice*. That is, human cognition is not a natural phenomenon, but a cultural-historical one. To take into consideration the role of the material in an account of human cognition is not enough. The material, as a theoretical category, cannot be taken for granted. We need to consider it from a historical

viewpoint. As Marx noted in a critique of Feuerbach's materialism, "As far as Feuerbach is a materialist, he does not deal with history, and as far as he considers history he is not a materialist" (Marx 1846/1998, p. 47).

As argued in this paper, the raw range of orienting-adjusting biological reactions we are born with is transformed into complex, historically constituted forms of sensing that are correlated with the historical material dimension of the world. As we live in society, interact with others, and participate in more or less specialized forms of training, the biological orienting-adjusting reactions undergo cultural transformation and are converted into complex historically constituted forms of sensation, leading to specific features of human development and the concomitant forms of cultural reflection (for a more detailed developmental account, see Radford 2013b). Thus, instead of being something that evolves naturally, cognition is considered to be a culturally and historically constituted, embodied, and materially sentient form of creatively responding to, acting, feeling, imagining, transforming, and making sense of the world.

The theoretical approach to cognition that I have outlined in this paper offers also a cultural-historical understanding of the conceptual. Concepts, I contended, are cultural codifications of human labor and, like all human labor, they are intrinsically multimodal. Their cultural codification is an institutional process of generalization by which they become endowed with what philosopher Evald Ilyenkov called *ideality*. In its dialectical materialist sense the *ideal*, Ilyenkov says, "is the [inter]subjective image of objective reality, i.e. reflection of the external world in the forms of man's (*sic*) activity, in the forms of his (*sic*) consciousness and will" (1977, p. 252). Concepts as ideal entities are not transcendental Kantian "things in themselves," nor are they ideas *in* the head. Ilyenkov continues:

> The ideal is not an individual, psychological fact, much less a physiological fact, but a socio-historical one… It exists in a variety of forms of man's (*sic*) social consciousness and will as the subject of the social production of material and spiritual life. (Ilyenkov 1977, p. 252)

In acquiring their ideality, concepts, as intersubjective generalized forms of action, keep their multimodal roots. But concepts are also multimodal in another sense. Since concepts are pure virtuality or pure possibility, their ontological nature is such that to become objects of thought and consciousness concepts have to be set into motion. They have to be actualized. Their multimodal nature reappears here, in their actualization in sensuous and material activity.

In the last part of the paper, I presented a short example that, I hope, gives an idea of the manner in which sensuous cognition may help us understand teaching–learning activity. Sensuous cognition, I argued, does not amount to claiming that our various senses come into play in classroom interaction. At this point, this is no more than a banal statement. The real question, I argued, is about understanding how, through classroom activity, our forms of sensing and reflecting are culturally transformed. The Grade 2 episode discussed in this paper provides us with some clues to answer this question. Indeed, our analysis reveals how, through the design of the task and its choice of questions—for example, questions about remote terms as well as decisions about whether or not terms belong to the sequence (e.g., Monique's term in our episode)—certain cultural mathematical rationality is *already highlighted*. Rather than natural, these questions are already loaded with a scientific rationality that invites the students to see the sequence and reflect about it in very historically constituted specific ways. For instance, Babylonian scribes and neo-Pythagoreans in the antiquity dealt with arithmetic sequences, and explored them in various ways; thus, Nicomachus of Geresa in the first century BC studied how polygonal numbers engendered other polygonal numbers (Nicomachus of Geresa 1938). Yet, the question of the calculation of the numerical value of remote figural terms—let alone the question of the general term—was not asked: to imagine the question and its answer required a *functional approach* and a rupture with a previous paradigmatic way of thinking (Radford 1995). It is in Diophantus' work in the third century BC (see Heath 1910) that we find already the question about the general term of a sequence (although, of course, not expressed through our modern symbolism, but through geometric deductions and natural language).

Now, even if culture and history already operate in a decisive manner in the design of the task by conveying historically evolved forms of thinking, imagining, and perceiving, the task in itself, generally speaking, is not enough to make the students aware of the sophisticated mathematical way of thinking that stresses numerical relationships between variables in a functional sense. For the students to become aware of, and participate in, such a mathematical way of thinking, the students need to engage in classroom forms of knowledge production and modes of human cooperation that—like Feuerbach's cherry-tree (see Sect. 3)—are not natural but have their own cultural history. And that is what the episode reveals. The teacher and the students *worked together* (see Fig. 5, Pics. 1–6) in order to achieve a perception of the terms of the sequence that was more sophisticated than the one the students achieved during the first 32 min of the math lesson, that is, before the teacher went to see the students' group. In the course of the teacher-and-students' joint work, the teacher and the students engaged in sensuous activity; they resorted to

gestures and words (see Fig. 5, Pics. 1–6) to open up a possibility to consider the terms of the sequence not as a heap of squares but as formed of squares distributed into rows. As a result, the students became progressively conscious of a spatial structure (*where* are the squares?) and a numerical one (*how many* squares?) whose coordination renders efficient a way of functional counting. The ensuing spatial–numerical way of perceiving the terms of the sequence is part of a culturally evolved mathematical way of thinking that can only become object of consciousness for the students through joint activity. In the course of this joint activity, the questions about remote terms made sense and a new form of theoretical perception replaced a naïve one. Thus, the students no longer needed to *see* the terms of the sequence to reflect on "remote" terms. The students-and-teacher's joint activity created the required support for an ontogenetic development out of which the students achieved a new form of *imagination* that makes possible "a departure from any immediate cognition of reality" (Vygotsky 1987, p. 349). This theoretical imagination is the support for the understanding of the concept of the general term of a sequence—something that we will never see; yet we can think of, talk about, and even calculate with.

As with all mathematical concepts, the concepts of remote terms and the general term of a sequence are infused with history and culture: it is the progressive and endless codified labor of previous generations (from Babylonian scribes, to Pythagoreans, to neo-Platonists, to abacists, to Mayas, to algebraists, etc.) who, for different reasons and in varied contexts, set themselves the task to investigate numbers and shapes, and to do so developed ways of thinking, imagining, and speaking that appear to the novice student as mere possibilities to reflect, act, and take action in the world. Since with the Hegelian dialectical materialist perspective concepts do not live apart from the sensuous experience that we make of the world, since, on the contrary, concepts are continuously enacted and actualized in our everyday life, they affect us unceasingly. They shape us and make us historical and cultural entities.

What, then, could the implications be for mathematics education? The sketched approach to cognition does not offer a recipe of how to teach or how to learn. It offers a cultural and historical dialectical materialist understanding of the senses, sensation and the material and conceptual worlds that can lead us to appraise in new ways the teaching and learning of mathematics. The approach is rather an invitation to understand the role of material culture and our senses in order to imagine and envision new ways in which to think the teaching and learning of mathematics. As biological material beings, we certainly have always been part of, and live in, material culture. Our mathematical practices have always been subsumed in the materiality around us—from the clay tablets of Sumerians to our sophisticated contemporary digital technologies. And if it is true that artifacts and material culture have been around for many decades in the teaching of elementary mathematics, through, for example, plastic blocks to count and colored shapes to touch, what is important to bear in mind is that, in the approach sketched here, these cultural artifacts are not merely devices that provide stimuli for cognitive development. They are part of cognition, which we see simultaneously as ideational and material.

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